Max-pressure signal control with cyclical phase structure

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1 Introduction

Intersections are a major bottleneck for urban networks. To optimize traffic signal timings, recent studies (Wongpiromsarn et al., 2012; Varaiya, 2013; Le et al., 2015) proposed max-pressure control techniques which use observed queue lengths to adaptively adjust signal timings. The key desirable properties of max-pressure control are *maximum stability*, i.e. max-pressure control is analytical proven to serve all demand if the demand could be served by *any* signal timing. Another nice property is *decentralized control*, in which the network-wide optimal solution can be found by a local computation at each intersection that depends only on the immediately upstream and downstream links.

Given these favorable characteristics, we seek to resolve a major practical issue that discourages implementation by city engineers. Specifically, drivers prefer traffic signals to follow a cyclical phase structure. Most work on max-pressure control (building off Wongpiromsarn et al., 2012; Varaiya, 2013) use a time step-based phase selection. (Although Le et al., 2015, has a signal cycle, the cycle length is fixed and phase durations can be arbitrarily small.) Although a non-cyclical phase selection may improve throughput, the limitations include potentially unbounded waiting times and the appearance of phases being "skipped" for waiting drivers. Due to the need to also serve pedestrians, and the desire to avoid complaints from drivers, the lack of a cyclical phase selection discourages implementation by city engineers.

The contributions of this paper are as follows: We modify Varaiya (2013)'s max-pressure control model and policy to follow a signal cycle with a maximum cycle length. The cycle length constraint restricts the size of the stable region, but we prove that the new max-pressure policy still has maximum stability (among signal timings that also follow the cycle length constraint). The resulting policy takes the form of model predictive control. Numerical results will compare delays and throughput for max-pressure control with and without the cyclical structure constraints.

2 Methodology

2.1 Network model

Consider a network $\mathcal{G} = (\mathcal{N}, \mathcal{A})$ with nodes $\mathcal{N}$ and directed links $\mathcal{A}$. The set of nodes is divided into junctions $\mathcal{N}_{\text{i}}$ and centroids $\mathcal{N}_{\text{z}}$. The set of links is divided into internal links $\mathcal{A}_{\text{i}}$ and entry links $\mathcal{A}_{\text{e}}$. Entry links connect a centroid to a junction. Internal links connect two junctions. All vehicles enter the network on an entry link and travel through the network until reaching their destination centroid. Route choice is modeled through exogenous turning proportions.

Consider discretized time. Assume without loss of generality that each link takes 1 time step to traverse at free flow. (Longer links can be divided into shorter segments.) Like previous work on max-pressure signal control (Varaiya, 2013; Le et al., 2015), we track the evolution of queue lengths per link using a store-and-forward queueing model. Let $x_{ij}(t)$ be the number of vehicles on link i waiting to move to link j. Link queues are separated by turning movements because different turning movements may not be actuated simultaneously during a traffic signal. Link queues evolve via conservation of flow. For internal links, flow conservation results in

$$x_{jk}(t+1) = x_{jk}(t) - y_{jk}(t) + \sum_{i \in \mathcal{A}} y_{ij}(t) r_{jk}(t) \tag{1a}$$

where $y_{jk}(t)$ is the flow of vehicles from j to k at time t, which is controlled by traffic signal actuation. Turning proportions $r_{jk}(t)$ determine the proportion of vehicles entering j that will next move to k. We assume that $r_{jk}(t)$ are independent identically distributed random variables with mean $\bar{r}_{jk}$. Flow conservation also applies to entry links, but entering flow is determined by the demand $d_i(t)$.

$$x_{ij}(t+1) = x_{ij}(t) - y_{ij}(t) + d_i(t) r_{ij}(t) \tag{1b}$$

We assume that for each entry link $i \in \mathcal{A}_{\text{e}}$, $d_i(t)$ for all t are independent identically distributed random variables with mean $\bar{d}_i$. We further assume that $d_i(t)$ has a maximum value $\tilde{d}_i$, which is reasonable because centroids are likely to have a physical capacity limitation. The queue length state $\mathbf{x}(t)$ forms a Markov chain with stochasticity due to the random demand $\mathbf{d}(t)$ and turning proportions $\mathbf{r}(t)$.

Intersection flows $y_{ij}(t)$ are controlled by the traffic signal activation. At each time step, a traffic signal phase is selected. Let $s_{ij}(t) \in \{0, 1\}$ indicate whether turning from (i, j) is permitted at time step t. Then $y_{ij}(t)$ is defined by

$$y_{ij}(t) = \min\left\{x_{ij}(t), s_{ij}(t) Q_{ij}\right\} \tag{2}$$

where Q_{ij} is the capacity of turning movement (i, j).

2.2 Traffic signal cycle

Each turning movement (i, j) is uniquely associated with one node n, where i is an incoming link and j is an outgoing link. Let $\mathcal{M}_n$ be the set of turning movements associated with n.

Let $\mathcal{P}_n$ be the ordered set of phases comprising the signal cycle for n. Let $p_n(t) \in [1, |\mathcal{P}_n|]$ be the phase number actuated at time step t for node n. Phases directly determine the turning movement activation $s_{ij}(t)$. Let $\xi_{ij}^p \in \{0, 1\}$ indicate whether phase p activates movement (i, j). Then

$$s_{ij}(t) = \xi_{ij}^{p_n(t)} \tag{3}$$

We assume that phases must be actuated in the specified order. To satisfy that, the phase at time step t is either the phase at $t-1$ or the next phase in the cycle:

$$p_n(t) \in \{p_n(t-1), p_n(t-1)+1\} \tag{4}$$

Furthermore, each phase must be actuated for at least one time step each cycle. Let C be the maximum cycle length. Let $c_n(t)$ be the duration of time since the signal cycle started for node n at time step t.

$$c_n(t+1) = \begin{cases} 1 & \text{if } p_n(t) = |\mathcal{P}_n| \text{ and } p_n(t+1) = 1 \\ c_n(t)+1 & \text{else} \end{cases} \tag{5}$$

To maintain a maximum cycle length, require that $c_n(t) \leq C$ for all t. Since each phase is actuated for at least one time step, the maximum cycle length imposes a constraint on phase actuation:

$$p_n(t) \geq |\mathcal{P}_n| - (C - c_n(t)) \tag{6}$$

2.3 Stable region

We first define stability mathematically. The network is stable if the number of vehicles in the network remains bounded in expectation. Equivalently, there exists a $\kappa < \infty$ such that

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \sum_{((i,j)\in\mathcal{A}^2)} \mathbb{E}\left[x_{ij}(t)\right] \leq \kappa \tag{7}$$

To define the stable region, let f_i be the average traffic volume for link i. For entry links,

$$f_i = \bar{d}_i \tag{8a}$$

For internal links, f_i can be determined by conservation of flow:

$$f_j = \sum_{i\in\mathcal{A}} f_i \bar{r}_{ij} \tag{8b}$$

The network can be stabilized if the average traffic volume can be served by some signal control. Equivalently, there must exist an average signal activation $\bar{s}_{ij}$ such that

$$f_i \bar{r}_{ij} \leq \bar{s}_{ij} Q_{ij} \tag{9}$$

Let $\mathcal{D}$ be the set of demand rates such that equations (8) and (9) hold. Let $\mathcal{D}^0$ be the interior of $\mathcal{D}$.

2.4 Max-pressure policy

First, define the weight for turning movement (i,j), $w_{ij}(t)$, as

$$w_{ij}(t) = x_{ij}(t) - \sum_{i \in \mathcal{A}} x_{jk}(t)\bar{r}_{jk} \tag{10}$$

This pressure has an intuitive interpretation of seeking to move vehicles from long queues to short queues. However, the precise form is analytically necessary for the stability proof.

Let $z_n^p(t) \in \{0,1\}$ indicate whether phase p is activated for node n at time t. Notice that $z_n^p(t-1)$ is exogenous and determined by $p_n(t-1)$, specifically

$$z_n^p(t-1) = \begin{cases} 1 & p = p_n(t-1) \\ 0 & \text{else} \end{cases} \tag{11}$$

To ensure that each phase is selected at most once,

$$\sum_{p=1}^{|\mathcal{P}_n|} z_n^p(t+\tau) = 1 \qquad \forall n \in \mathcal{N}_{\mathrm{i}}, \forall \tau \in [0, \mathcal{T}-1] \tag{12}$$

Since phases proceed in order, $z_p(t)$ is constrained by $z_p(t-1)$. Equation (4) can be written as the following constraint:

$$z_n^p(t+\tau) \leq z_n^p(t+\tau-1) + z_n^{p-1}(t+\tau-1) \qquad \forall n \in \mathcal{N}_{\mathrm{i}}, \forall \tau \in [0, \mathcal{T}-1] \tag{13}$$

which requires that $z_n^p(t) = 1$ only if phase p or $p-1$ was active at time $t-1$. Since phases follow a cycle, when $p = 1$ phase $p-1$ refers to phase $|\mathcal{P}_n|$.

Notice also that $c_n(t-1)$ is exogenous, and determines the number of time steps remaining before the maximum cycle length is reached. Let $\varphi_n(t) \in \{0,1\}$ indicate whether the cycle restarts at time t for node n.

$$\varphi_n(t+\tau) \leq z_n^1(t+\tau) - z_n^{|\mathcal{P}_n|}(t+\tau-1) \qquad \forall n \in \mathcal{N}_{\mathrm{i}}, \forall \tau \in [0, \mathcal{T}-1] \tag{14}$$

which admits $\varphi_n(t) = 1$ only when the phase at node n switches from $|\mathcal{P}_n|$ to phase 1 from time $t-1$ to t. Let $c_n(t)$ be the number of time steps since the cycle was started, as defined by equation (5):

$$c_n(t+\tau) = \begin{cases} c_n(t+\tau-1) + 1 & \varphi_n(t+\tau) = 0 \\ 1 & \varphi_n(t+\tau) = 1 \end{cases} \qquad \forall n \in \mathcal{N}_{\mathrm{i}}, \forall \tau \in [0, \mathcal{T}-1] \tag{15}$$

The maximum cycle length is enforced by the constraint

$$c_n(t+\tau) \leq C_n \qquad \forall n \in \mathcal{N}_{\mathrm{i}}, \forall \tau \in [0, \mathcal{T}-1] \tag{16}$$

The final constraint is to relate $s_{ij}(t+\tau)$ with the phase selection:

$$s_{ij}(t+\tau) = \sum_{p=1}^{|\mathcal{P}_n|} z^p - n(t)\xi_{ij}^p \qquad \forall n \in \mathcal{N}_{\mathrm{i}}, \forall \tau \in [0, \mathcal{T}-1] \tag{17}$$

The max-pressure policy is found by solving the following integer linear program:

$$\max \quad \frac{1}{T}\sum_{\tau=0}^{\mathcal{T}-1} \sum_{(i,j)\in\mathcal{A}^2} s_{ij}(t+\tau) Q_{ij} w_{ij}(t) \tag{18}$$

$$\text{s.t.} \quad (12)\text{–}(17)$$

The optimal solution to problem (18) at time t, $\mathbf{s}^\star(t)$, is actuated at time step t. The remainder of the horizon of the optimal solution, $\mathbf{s}^\star(t+\tau)$ for $t \in [1, \mathcal{T}]$, is included only for planning purposes and discarded after actuating the solution at time t.

Proposition 1. *If $\bar{\mathbf{d}} \in \mathcal{D}^0$, then the max-pressure policy is stabilizing. If $\bar{\mathbf{d}} \notin \mathcal{D}$, then no signal timing is stabilizing.*

3 Conclusions

This paper developed a max-pressure control with a cyclical phase structure. A store-and-forward queueing model similar to Varaiya (2013) was created, but with the addition of cycle length and phase selection constraints. The set of demand rates that could be served by any signal timing was analytically described, and used to prove that the proposed max-pressure policy has maximum stability. Due to the signal cycle structure, the max-pressure control takes the form of a model predictive control.

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